

PINCHUK MAPS AND FUNCTION FIELDS

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ABSTRACT. All counterexamples of Pinchuk type to the strong real Jacobian conjecture are shown to have rational function field extensions of degree six with no nontrivial automorphisms.

1. INTRODUCTION

The unresolved Jacobian Conjecture asserts that a polynomial map $F : k^n \rightarrow k^n$, where k is a field of characteristic zero, has a polynomial inverse if its Jacobian determinant, $j(F)$, is a nonzero element of k . For $k = \mathbb{R}$ the similar Strong Real Jacobian Conjecture was that F is bijective if it is a polynomial map and $j(F)$, though perhaps not a constant, vanishes nowhere on \mathbb{R}^n . However, Sergey Pinchuk exhibited a family of counterexamples for $n = 2$, now usually called Pinchuk maps.

Any polynomial endomorphism of \mathbb{R}^n , whose Jacobian determinant does not vanish identically, induces a finite algebraic extension of rational function fields, and if it is bijective it is easy to show that the extension is of odd degree, with no nontrivial automorphisms. This extension is investigated for a previously well studied Pinchuk map. A primitive element is found, its minimal polynomial is calculated, and the degree (6) and automorphism group (trivial) of the extension are determined. That generalizes to any Pinchuk map $F(x, y) = (P(x, y), Q(x, y))$ defined over any subfield k of \mathbb{R} . Although F is generically two to one as a polynomial map of \mathbb{R}^2 to \mathbb{R}^2 , the degree of the associated extension of function fields $k(P, Q) \subset k(x, y)$ is 6 and $k(x, y)$ admits no nontrivial automorphism that fixes all the elements of $k(P, Q)$ (Theorem 1).

2. A SPECIFIC PINCHUK MAP

Pinchuk maps are certain polynomial maps $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that have an everywhere positive Jacobian determinant $j(P, Q)$, and are not injective [10]. The polynomial $P(x, y)$ is constructed by defining $t = xy - 1$, $h = t(xt + 1)$, $f = (xt + 1)^2(t^2 + y)$, $P = f + h$. The polynomial Q varies for different Pinchuk maps, but always has the form $Q = q - u(f, h)$, where $q = -t^2 - 6th(h + 1)$ and u is an auxiliary polynomial in f and h , chosen so that $j(P, Q) = t^2 + (t + f(13 + 15h))^2 + f^2$.

The specific Pinchuk map used here is one introduced by Arno van den Essen via an email to colleagues in June 1994. It is defined [4] by choosing

$$(1) \quad u = 170fh + 91h^2 + 195fh^2 + 69h^3 + 75fh^3 + \frac{75}{4}h^4.$$

The total degree in x and y of P is 10 and that of Q is 25.

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The image, multiplicity and asymptotic behavior of F were studied in [1, 2, 3]. Its asymptotic variety, $A(F)$, is the set of points in the image plane that are finite limits of the value of F along curves that tend to infinity in the (x, y) -plane [8, 9]. It may alternatively be defined as the set of points in the image plane that have no neighborhood with a compact inverse image under F [5, 6, 7]. It is a topologically closed curve in the image (P, Q) -plane and is the image of a real line under a bijective polynomial parametrization. It is depicted below using differently scaled P and Q axes. It intersects the vertical axis at $(0, 0)$ and $(0, 208)$. Its leftmost point is $(-1, -163/4)$, and that is the only singular point of the curve.

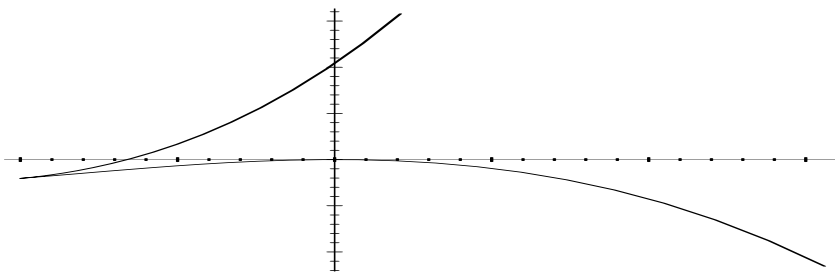


FIGURE 1. The asymptotic variety of the Pinchuk map F .

The image plane coordinates of points on $A(F)$ satisfy [3] the irreducible polynomial equation

$$(Q - (345/4)P^2 - 231P - 104)^2 = (P + 1)^3(75P + 104)^2.$$

The point $(-104/75, -18928/375)$ (approximately $(-1.38, -50.47)$) also satisfies this equation, and so lies in the Zariski closure of $A(F)$, but does not lie on the curve $A(F)$ itself. So $A(F)$ is an irreducible semi-algebraic set, but not an actual algebraic variety.

The points $(-1, -163/4)$ and $(0, 0)$ of $A(F)$ have no inverse image under F , all other points of $A(F)$ have one inverse image, and all points of the image plane not on $A(F)$ have two.

3. MINIMAL POLYNOMIAL CALCULATION

This paragraph is a summary of some key facts from previously cited work on F . A general level set $P = c$ in the (x, y) -plane has a rational parametrization. Specifically, for any real c that is not -1 or 0 , the equations

$$x(h) = \frac{(c - h)(h + 1)}{(c - 2h - h^2)^2}$$

$$y(h) = \frac{(c - 2h - h^2)^2(c - h - h^2)}{(c - h)^F 2},$$

define a rational map pointwise on a real line with coordinate h , except where a pole occurs. The use of h as a parameter and the equality $P = c$ are consistent: on substitution the expression $h(x(h), y(h))$ simplifies to h , and $P(x(h), y(h))$ to

c. There is always a pole at $h = c$ and $Q(x(h), y(h))$ tends to $-\infty$ as the pole is approached from either side. Also, $Q(x(h), y(h))$ tends to $+\infty$ as h tends to $+\infty$ or to $-\infty$. If $c > -1$, there are two additional poles at $h = -1 \pm \sqrt{1+c}$ and $Q(x(h), y(h))$ tends to a finite asymptotic value at each of these poles as the pole is approached from either side. For example, if $c = 3$, there are poles at -3 , 1 , and 3 , and the corresponding values of Q are $14965/4$, $-4235/4$, and $-\infty$, respectively. In such cases the asymptotic values are distinct and are the values of Q at the two points of intersection of the vertical line $P = c$ and $A(F)$ in the (P, Q) -plane. The level sets $P = c$ are disjoint unions of their connected components, which are curves that are smooth (because of the Jacobian condition) and tend to ∞ in the (x, y) -plane at both ends. The number of curves is two if $c < -1$, and four if $-1 < c \neq 0$. Even the two exceptional values fit this pattern, although they require different rational parametrizations, with $P = -1$ consisting of four curves and $P = 0$ of five.

F is not birational, because it is generically two to one. Throughout this section, consider only the coefficient field $k = \mathbb{R}$. To begin the exploration of the field extension $k(P, Q) \subset k(x, y)$, rewrite the parametrization above in terms of f and h , using the relations $P = c$ and $P = f + h$ to obtain

$$x = f(h+1)(f-h-h^2)^{-2}$$

$$y = (f-h-h^2)^2(f-h^2)f^{-2},$$

which are identities in $k(x, y)$ (and so $k(x, y) = k(f, h)$). It follows that $xy = (h+1)(f-h^2)/f$, $t = xy - 1 = [(h+1)(f-h^2) - f]/f = [fh - h^2 - h^3]/f = (h/f)[f - h(h+1)]$, $q = -t^2 - 6th(h+1) = -h^2f^{-2}\{[f - h(h+1)]^2 + 6(h+1)[f - h(h+1)]f\}$. In fact,

$$\begin{aligned} q &= -h^4(h+1)^2/f^2 + [2h^3(h+1) + 6h^3(h+1)^2]/f + [-h^2 - 6h^2(h+1)] \\ &= -h^4(h+1)^2/f^2 + h^3(h+1)(6h+8)/f - h^2(6h+7). \end{aligned}$$

Using that equation, the definition $Q = q - u(f, h)$, and equation 1 for u one can express Q in terms of f and h alone. Clearing denominators $f^2Q = f^2q - f^2u$, or, arranged by powers of f ,

$$\begin{aligned} f^2Q &= -h^4(h+1)^2 \\ &\quad + f[h^3(h+1)(6h+8)] \\ &\quad + f^2[-h^2(6h+7) - 91h^2 - 69h^3 - (75/4)h^4] \\ &\quad + f^3[-170h - 195h^2 - 75h^3]. \end{aligned}$$

Now substitute $P-h$ for f and collect in powers of h to obtain a polynomial relation

$$(197/4)h^6 + \cdots + (2PQ - 170P^3)h - P^2Q = 0.$$

Let $R(T)$ be the corresponding polynomial in T with root h .

Straightforward computations show that

$$\begin{aligned} R(T) &= (197/4)T^6 + (104 - (363/2)P)T^5 + (63 - 421P + (825/4)P^2)T^4 \\ &\quad + (-306P + 510P^2 - 75P^3)T^3 + (-Q + 412P^2 - 195P^3)T^2 \\ &\quad + (2PQ - 170P^3)T - P^2Q. \end{aligned}$$

The coefficient of each power of T is a polynomial in P and Q with rational coefficients, and has total degree in P and Q at most 3. Since the leading coefficient of $R(T)$ is a real constant, the fact that $R(h) = 0$ shows that h is integral over

$k[P, Q]$. The identity $R(h) = 0$ in $\mathbb{Q}[x, y,]$ was successfully checked by a computer symbolic algebra program.

Let $m(T)$ be the polynomial in $k[P, Q][T]$, T an indeterminate, which has leading coefficient 1 and satisfies $m(h) = 0$ in $k[x, y]$, and which is of minimal degree. Clearly m is irreducible in both $k[P, Q][T]$ and $k[P, Q, T]$, hence by the Gauss Lemma, in $k(P, Q)[T]$. That implies that m is also of minimal degree over $k(P, Q)$, that m divides any polynomial in $k[P, Q][T]$ with h as a root, and that m is unique.

Factorize F as $\mathbb{R}^2 \rightarrow V(m) \rightarrow \mathbb{R}^2$, where the first map is (P, Q, h) , $V(m)$ is the zero locus of m in \mathbb{R}^3 , and the second map is the projection onto the first two components. The first map is birational ($k(P, Q)(h) = k(x, y)$) and the second is finite by integrality. If $w = (w_1, w_2)$ is a point of the (P, Q) -plane, call r a root of m over w , if $m(w_1, w_2, r)$ (write as $m(w, r)$) is zero.

Lemma 1. *For generic w , m has exactly two real roots over w , they are simple and distinct, w has exactly two inverse images v and v' under F , and the real roots of m over w are $r = h(v)$ and $r' = h(v')$.*

Proof. Take a bi-regular isomorphism from a Zariski open subset O of the (x, y) -plane to a Zariski open subset of $V(m)$. The image is a nonsingular surface S . In the usual (strong) topology it has a finite number of connected components that are open subsets of S and of $V(m)$. Take the union of i) the image of the complement of O under F , ii) the projection of the complement of S , and iii) $A(F)$. From the Tarski-Seidenberg projection property and other basic tools of real semi-algebraic geometry, the union is semi-algebraic of maximum dimension 1. Take w in the complement of the Zariski closure of that union. Any root that lies over w is a nonsingular point of $V(m)$ (by construction), and so is a simple, not multiple, real root. There are exactly two points, say v and v' , that map to w under F . Their images under h , r and r' , lie over w . By construction (w, r) and (w, r') lie in S , and v and v' lie in O . Hence r and r' are distinct. No point in the complement of O can map to (w, r) or (w, r') (by construction), and v and v' are the only points in O that do so. \square

Corollary 2. *The T -degree of m is even.*

Proof. The complex roots over w that are not real occur in complex conjugate pairs. \square

Let m_0 be the term of $m(T)$ of degree 0 in T . Clearly, $m_0 \in k[P, Q]$ is not the zero polynomial. Since $m(T)$ divides $R(T)$, m_0 divides P^2Q . Since the T -degree of m is even, $m_0(w)$ is the product of all the roots, real and complex, of m over w , for any w in the (P, Q) -plane.

Proposition 3. *m_0 is a positive constant multiple of $-P^2Q$.*

Proof. $F(1, 0) = (0, -1)$ and $h(1, 0) = 0$, so $m_0(0, -1) = 0$. This shows that P must divide m_0 , for otherwise $m_0(0, -1)$ would be nonzero. Next, $F(1, 1) = (1, 0)$ and $h(1, 1) = 0$, so $m_0(1, 0) = 0$. So Q divides m_0 . Next, consider the union of the vertical lines $P = c$ in the (P, Q) -plane, for $2 < c < 4$. At least one such line must contain a point w that is generic in the sense of Lemma 1, for otherwise there would be an open set of nongeneric points. Choose such a c , and note that all but finitely many points of the line $P = c$ are generic. The level set $P = c$ in the (x, y) -plane has the rational parametrization by h already described, with a pole

at $h = c$, at which Q tends to $-\infty$. Take a point $w = (c, d)$ with d negative and sufficiently large. Then w will be generic and its two inverse images under F will have values of h that approach c as d tends to $-\infty$, one value of h less than c and the other greater. The product of all the roots of m over w will be positive, since the nonreal roots occur as conjugate pairs. Since P is positive and Q negative, the numerical coefficient of m_0 must be negative, regardless of whether m_0 is exactly divisible by P or by P^2 . Finally, make a similar argument for a suitable line $P = c$, with $-4 < c < -2$. Consideration of signs shows that m_0 must be divisible by P^2 , which yields the desired conclusion. \square

Corollary 4. *The T -degree of m is not 2.*

Proof. If m has degree 2 and w is generic, then $m_0(w)$ is exactly the product of the two real roots of m over w . But for the last two examples considered in the previous proof, the product tends to c^2 as h tends to c , whereas $m_0(w)$ is unbounded. \square

Proposition 5. $R(T) = (197/4)m(T)$.

Proof. As $m(T)$ is a nonconstant divisor of $R(T)$ of even T -degree not equal to 2, it remains only to show that the degree of m in T is not 4. Suppose to the contrary that $(m_4T^4 + m_3T^3 + m_2T^2 + m_1T + m_0)(d_2T^2 + d_1T + d_0) = R(T)$, where the first factor is $m(T)$, the second is a polynomial $D(T)$ of degree 2 in T , and all the coefficients shown are in $k[P, Q]$. Note that the equation is in $k[P, Q][T] = k[P, Q, T]$, where P , Q , and T are simply algebraically independent variables. Equating leading and constant terms on both sides, one finds that $m_4 = 1$, $d_2 = 197/4$ and d_0 is a positive constant. The coefficient d_1 must also be constant. For if not, $j = \deg^t(d_1) > 0$, where \deg^t temporarily denotes the total degree in P and Q . As noted earlier, that \deg^t is at most 3 for every coefficient of $R(T)$. Starting with $\deg^t(m_0) = 3$ and equating in turn terms of T -degree 1 through 4 on both sides of the equation assumed for $R(T)$, one readily finds that $\deg^t(m_4) = 3 + 4j$. But $m_4 = 1$, a contradiction. Thus $D(T)$ has constant coefficients. Next, set $P = 0$ in $R(T)$, obtaining $(197/4)T^6 + 104T^5 + 63T^4 - QT^2$. Further setting $Q = 0$, one finds that the resulting polynomial in T alone factors as $T^4((197/4)T^2 + 104T + 63)$. Clearly D must be exactly the quadratic factor shown. But if $D(T)$ divides $R(T)$, setting $P = 0$ implies it must also divide $-QT^2$, which is absurd. That contradiction shows that the original assumption to the contrary, that m has T -degree 4, is false. \square

Recall that only the coefficient field $k = \mathbb{R}$ is considered in this section.

Corollary 6. *The field extension $\mathbb{R}(P, Q) \subset \mathbb{R}(x, y)$ is of degree 6.*

Proof. Clear. \square

4. AUTOMORPHISMS OF THE EXTENSION

Again, assume throughout this section that $k = \mathbb{R}$. Let $Z = F^{-1}(A(F))$. For any $(x, y) \notin Z$ there is a unique different point $(x', y') \notin Z$ with the same image under F . That defines an involution τ of $\mathbb{R}^2 \setminus Z$, that is a semi-algebraic real analytic diffeomorphism. Suppose φ is an automorphism of $k(x, y)$ that is not the identity but fixes every element of $k(P, Q)$. If φ preserves h , then it also preserves x and y , since they are rational functions of P and h , namely

$$(2) \quad x = \frac{(P-h)(h+1)}{(P-2h-h^2)^2} \text{ and } y = \frac{(P-2h-h^2)^2(P-h-h^2)}{(P-h)^2}.$$

So $\varphi(h) = h' \neq h$ and, furthermore, by the identity principle for rational functions over \mathbb{R} , they cannot be equal on any nonempty open subset of the (x, y) -plane on which h' is defined. Since φ preserves both components of F , the fact that its geometric realization cannot be the identity even locally means that it must be τ wherever both are defined. That implies that there can be at most one such a nonidentity automorphism φ . If it exists, then the rational function h' is analytic at any point $(x, y) \notin Z$, since $h'(x, y) = h(x', y') = h(\tau(x, y))$.

That reduces the question of the existence of φ to the following one. Is h' , a well defined real analytic function on the complement of Z in the (x, y) -plane, in fact a real rational function?

Lemma 7. *There are three component curves of the level set $P = 0$ on which h is nonconstant and vanishes nowhere. On those curves $Q = Q(h) = h^2((197/4)h^2 + 104h + 63)$ and is everywhere positive. There is one point of $Z = F^{-1}(A(F))$ on the three curves. At all of their other points, h' satisfies both $h' \neq h$ and $Q(h') = Q(h)$.*

Proof. Set $P = 0$ in equation 2. The resulting rational functions $x(h), y(h)$ are defined everywhere except at $h = -2$ and $h = 0$. That yields three curves parametrized by h . Since $h(x(P, h), y(P, h))$ simplifies to h , $h(x(h), y(h)) = h$ and hence h assumes every real value exactly once on these curves, except that -2 and 0 are never assumed. Since every level set $P = c$ is a finite disjoint union of closed connected smooth curves unbounded at both ends, each of the three curves is a connected component of $P = 0$.

Set $P = 0$ in the identity $R(h) = 0$, obtaining $(197/4)h^6 + 104h^5 + 63h^4 - Qh^2 = 0$. On the curves, $h \neq 0$, so the claimed formula for Q follows. Furthermore, Q is positive there, since $(197/4)h^2 + 104h + 63$ has negative discriminant.

So F maps points on the three curves to the positive Q -axis. Routine calculation of the derivative of Q shows that Q is monotonic decreasing for $h < 0$ and monotonic increasing for $h > 0$. Considering the graph of Q , one concludes that every positive real is the value of Q exactly twice, for nonzero values of h of opposite signs. Those values of h all correspond to unique points of the curves, except $h = -2$. As $Q(-2) = 208$, the point $(0, 208)$, which is the only point of $A(F)$ on the positive Q -axis, has as its unique inverse under F the point $(x(h'), y(h'))$, where h' is the positive real satisfying $Q(h') = 208$. \square

Remark. To clarify, there are two additional component curves of the level set $P = 0$. On them $h = 0$ identically. They have a rational parametrization by t with a pole at $t = 0$ and $Q = -t^2$ is everywhere negative on them.

Lemma 8. *Let h' be any real rational function of h that satisfies $Q(h') = Q(h)$ for infinitely many values of $h \in \mathbb{R}$. Then $h' = h$.*

Proof. Suppose $h' = a/b$ for polynomials $a, b \in \mathbb{R}[h]$, of respective degrees r, s , with no common divisor. From $Q(h') = Q(h)$ one obtains

$$(3) \quad a^2((197/4)a^2 + 104ab + 63b^2) = b^4h^2((197/4)h^2 + 104h + 63),$$

a polynomial equality that is true for all real h . Since a/b tends to ∞ as h does, $r > s$. Counting degrees $4r = 4s + 4$, so $r = s + 1$. The factor in parentheses on the left is quadratic and homogeneous in a and b and has negative discriminant, so it is zero for real a and b only if both are zero. But that cannot occur for any real h , for then a and b would have a common root, hence a common divisor. Thus that

factor has only complex roots. It follows that h^2 divides a^2 , and so h divides a . That means that a and bh share a root, each having h as a factor. As the quadratic factor in parentheses on the right is not zero for any real h , any further real roots (at $h = 0$ or not) in the two sides of equation 3 would be shared by a and b . Again, that is not possible, and therefore a has no further real roots and all the roots of b are complex. In particular s , the degree of b , must be even. No complex root of b can be a root of a , as that would imply a common real irreducible quadratic factor. So it must be a root of the parenthetical factor on the left. Counting roots with multiplicities, $2r = 2s + 2 \geq 4s$, so $s \leq 1$. Since s is even, it must be 0, and so $h' = \lambda h$ for a nonzero $\lambda \in \mathbb{R}$. Then for any fixed $h \neq 0$, $Q(\lambda^i h) = Q(h)$ is independent of $i > 0$, and so λ has absolute value 1. It cannot be -1 , because $Q(h) - Q(-h) = 208h^3$. \square

Proposition 9. *The group of automorphisms of the field extension $\mathbb{R}(P, Q) \subset \mathbb{R}(x, y)$ is trivial.*

Proof. If the group contains a nontrivial automorphism φ , then $h' = \varphi(h) = h(x', y')$ (see above) belongs to $\mathbb{R}(x, y) = \mathbb{R}(P, h)$. As $\mathbb{R}(P, h)$ is a rational function field in two algebraically independent elements over \mathbb{R} , the restriction of h' to the level set $P = 0$ must either be generically undefined (uncanceled P in the denominator) or a rational function of h . The first case is ruled out by Lemma 7, which also contradicts Lemma 8 in the second case. \square

5. ALL PINCHUK MAPS

From a geometric point of view, any two different Pinchuk maps are very closely related. More specifically, if $F_1 = (P, Q_1)$ and $F_2 = (P, Q_2)$ are Pinchuk maps then they have the same first component, P , and their second components satisfy $Q_2 = Q_1 + S(P)$ for a polynomial S in one variable with real coefficients [3]. As maps of \mathbb{R}^2 to \mathbb{R}^2 , therefore, they differ only by a triangular polynomial automorphism of the image plane. So all Pinchuk maps are generically two to one, and their asymptotic varieties have algebraically isomorphic embeddings in the image plane.

Let F be the same Pinchuk map as before. It is defined over \mathbb{Q} . In fact, not only do P and Q have rational coefficients, but so do h and all terms of the minimal polynomial m for h . Let k be any subfield of \mathbb{R} , including the possibilities $k = \mathbb{Q}$ and $k = \mathbb{R}$. Then the powers h^i for $i = 0, \dots, 5$ form a basis for $k(x, y)$ as a vector space over $k(P, Q)$ and the field extension is of degree 6.

Proposition 10. *Let F_1 and F_2 be Pinchuk maps defined over k and connected by S . Then S is uniquely determined and its coefficients belong to k .*

Proof. P is transcendental over \mathbb{R} , so S is unique. Let $c \in \mathbb{Q}$ with $c \neq 0$ and $c \neq -1$. Choose $h \in \mathbb{Q}$ that is not a pole of the previously described rational parametrization $x(h), y(h)$ of the level set $P = c$. Since both $x(h)$ and $y(h)$ have formulas in $\mathbb{Q}(h, c)$, the real number $S(c) = Q_2(x(h), y(h)) - Q_1(x(h), y(h))$ actually is in k . The coefficients of S can be reconstructed, using rational arithmetic, from its values at any $j > \deg S$ such points c , and so are in k . \square

Corollary 11. *Any two Pinchuk maps defined over k have one and the same field extension over k .*

Proof. $k(P, Q_1) = k(P, Q_2)$ follows from the even stronger consequence $k[P, Q_1] = k[P, Q_2]$. \square

Theorem 1. *Let F be any Pinchuk map and let k be \mathbb{R} or any subfield of \mathbb{R} containing the coefficients of F . Although F is generically two to one as a polynomial map of \mathbb{R}^2 to \mathbb{R}^2 , the degree of the associated extension of function fields over k is six. Furthermore, the extension has no automorphisms other than the identity.*

Proof. The conclusions have already been drawn for the earlier specific Pinchuk map $F = (P, Q)$ and for $k = \mathbb{R}$ (Corollary 6 and Proposition 9). Both Pinchuk maps have the same function field extension over k , so it has degree six. And any nontrivial automorphism defined over k defines one over \mathbb{R} , since the \mathbb{R} -linearly extended automorphism preserves P , Q , and \mathbb{R} . \square

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